On the spectrum of a class of quantum models

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Abstract – The spectrum of any quantum model which eigenvalue equation reduces to a three-term recurrence, such as a displaced harmonic oscillator, the Jaynes-Cummings model, the Rabi model, and a generalized Rabi model, can be determined as zeros of a corresponding transcendental function $F(x)$. The latter can be analytically determined as an infinite series defined solely in terms of the recurrence coefficients.

Introduction. – The present work deals with a class $\mathcal{R}$ of quantum models described by a Hamiltonian $\hat{H}$ characterized in that the eigenvalue equation

$$\hat{H}\phi = E\phi \tag{1}$$

in the Bargmann Hilbert space of analytical functions $\mathcal{B}$ [1, 2] reduces to a *three-term difference equation*

$$c_{n+1} + a_n c_n + b_n c_{n-1} = 0 \qquad (n \geq 0). \tag{2}$$

Here $\{c_n\}_{n=0}^{\infty}$ are the sought expansion coefficients of a physical state described by an entire function

$$\phi(z) = \sum_{n=0}^{\infty} c_n z^n. \tag{3}$$

The recurrence coefficients a_n and b_n are functions of model parameters and we require that $b_n \neq 0$. In what follows, models of the class $\mathcal{R}$ are assumed to satisfy $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n/a_n = 0$. The recurrence coefficients may have either finite limits

$$a_n \to a, \qquad b_n \to b \qquad (n \to \infty) \tag{4}$$

(a *Poincaré difference equation* [3]), or a power-like dependence $a_n \sim an^{\delta}$, $b_n \sim bn^{v}$.

The conventional boson annihilation and creation operators $\hat{a}$ and $\hat{a}^{\dagger}$ satisfying commutation relation $[\hat{a}, \hat{a}^{\dagger}] = 1$ are represented in the Bargmann Hilbert space $\mathcal{B}$ as [1, 2]

$$\hat{a} \to \frac{\partial}{\partial z}, \qquad \hat{a}^{\dagger} \to z. \tag{5}$$

Therefore, the class $\mathcal{R}$ comprises among others single-mode boson models represented by

$$\hat{H} = A_1 \hat{a}^{\dagger}\hat{a} + A_2 \hat{a}^{\dagger} + A_3 \hat{a} + A_4, \tag{6}$$

where A_j are in general matrix coefficients. Indeed, on comparing the same powers of z on both sides of (1), the consecutive $\hat{a}^{\dagger}\hat{a}$, $\hat{a}^{\dagger}$, and $\hat{a}$ terms in (6) lead to nc_n, c_{n-1}, and $(n+1)c_{n+1}$ terms in the recurrence (2), respectively. The eigenvalue equation (1) then reduces to a coupled system of ordinary differential equations [1, 4, 5]. The model Hamiltonian (6) encompasses many prominent examples, such as a displaced harmonic oscillator [1], the Rabi model [1,6,7], the Jaynes and Cummings (JC) model proposed as an approximation to the Rabi model [8], and a generalized Rabi model [9]. The Rabi model describes the simplest interaction between a cavity mode with a bare frequency ω and a two-level system with a bare resonance frequency ω_0. The model is characterized by the Hamiltonian [1, 6, 7, 9],

$$\hat{H}_R = \hbar\omega \hat{a}^{\dagger}\hat{a} + \lambda\sigma_1(\hat{a}^{\dagger} + \hat{a}) + \mu\sigma_3, \tag{7}$$

where λ is a coupling constant, and $\mu = \hbar\omega_0/2$. In what follows we assume the standard representation of the Pauli matrices σ_j and set the Planck constant $\hbar = 1$.

Our main results is that the spectrum of any quantum model from $\mathcal{R}$ can be obtained as zeros of a transcendental function defined by infinite series

$$F(x) \equiv a_0 + \sum_{k=1}^{\infty} \rho_1 \rho_2 \dots \rho_k \tag{8}$$

solely in terms of the coefficients of the three-term recurrence relation in eq. (2) [10]:

$$\begin{aligned} \rho_1 &= -\frac{b_1}{a_1}, \quad \rho_l = u_l - 1, \quad u_1 = 1, \\ u_l &= \frac{1}{1 - u_{l-1} b_l/(a_l a_{l-1})}, \quad l \geq 2. \end{aligned} \tag{9}$$

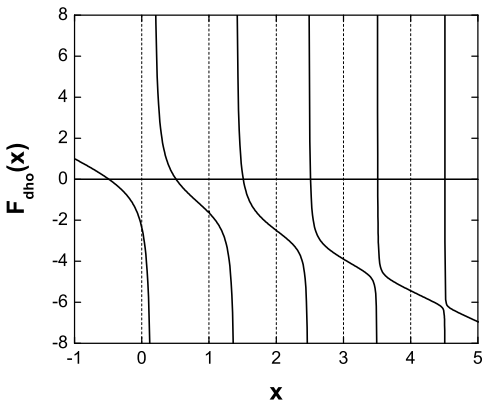

Fig. 1: F_{dho} as a function of $x = E/\omega$ for $\kappa = \lambda/\omega = 0.7$, $\Delta = \mu/\omega = 0$, in the case when the spectrum of the Rabi model reduces to that of a displaced harmonic oscillator. In agreement with the exact analytic formula (15), the zeros of $F_{dho}(x)$ satisfy $x_l = l - 0.49$.

The function $F(x)$ is different from the G-functions of Braak [9]. The latter have only been obtained in the case of the Rabi model by making explicit use of the parity symmetry. In contrast to Braak's result [9], our $F(x)$ can be straightforwardly defined for any model from $\mathcal{R}$. The definition of $F(x)$ does not require either a discrete symmetry or to solve the three-term difference equation (2) explicitly. All what is needed to determine $F(x)$ is an explicit knowledge of the recurrence coefficients a_n and b_n. This brings about a straightforward numerical implementation [11] and results in a great simplification in determining the spectrum. The proof of principle is demonstrated in fig. 1 in the case of exactly solvable displaced harmonic oscillator [1]. The ease in obtaining the spectrum is of importance regarding recent experimental advances in preparing (*ultra*)strongly interacting quantum systems [12–16], which can no longer be reliably described by the exactly solvable JC model [8], and wherein only the full quantum Rabi model can describe the observed physics.

Proof of the main result. – In earlier studies [1, 4, 5, 9], largely motivated by the Frobenius method [17] of solving differential equations, the $n = 0$ part of the recurrence (2) was taken as an initial condition. Given the requirement of analyticity [$c_{-k} \equiv 0$ for $k > 1$; *cf.* eq. (3)], the three-term recurrence (2) degenerates for $n = 0$ into an equation involving mere *two terms* and imposes that

$$c_1/c_0 = -a_0. \tag{10}$$

By solving the recurrence upwardly, one as a rule arrives at the expansion coefficients of *singular* functions *outside* the physical Bargmann Hilbert space $\mathcal{B}$ [1,4,17]. Only at a particular discrete set of parameter values that correspond to the spectrum of a model the functions would belong to $\mathcal{B}$ [1, 5]. Our approach differs from earlier studies [1, 5, 9] in that our focus is initially on the truly *three-term* part of the recurrence (2) for $n \geq 1$. This would enable us to remain in the physical Bargmann Hilbert space $\mathcal{B}$ for any value of physical parameters. The remaining $n = 0$ part of the recurrence (2), which reduces to a relation between c_1 and c_0, then becomes the *boundary condition* for $\phi \in \mathcal{B}$ defined by the $n \geq 1$ solutions of the recurrence (2).

In virtue of the Perron and Kreuser generalizations (Theorems 2.2 and 2.3(a) in ref. [3]) of the Poincaré theorem (Theorem 2.1 in ref. [3]), the difference equation (2) (considered for $n \geq 1$) possesses in our case *two* linearly independent solutions:

- (i) a *dominant* solution $\{d_j\}_{j=0}^{\infty}$ and
- (ii) a *minimal* solution $\{m_j\}_{j=0}^{\infty}$.

The respective solutions differ in their limiting behavior c_{n+1}/c_n in the limit $n \to \infty$. The the so-called *minimal* solution guaranteed by Perron's and Kreuser's theorems satisfies [3, 18]

$$\lim_{n\to\infty} m_{n+1}/m_n \equiv 0. \tag{11}$$

On substituting the minimal solution of (2) for the c_n's in eq. (3), $\phi(z)$ automatically becomes an *entire* function. In what follow, the entire function generated by the minimal solution of the $n \geq 1$ part of (2) will be denoted as $\phi_m(z)$. As demonstrated in the case of the Rabi model below, $\phi_m(z)$ belongs to $\mathcal{B}$ *irrespective* of the value of the model parameters.

Now it is important to realize that only the ratios of subsequent terms m_{n+1}/m_n of the *minimal* solution (and hence not of the *dominant* solution) are related to infinite continued fractions (*cf.* Theorem 1.1 due to Pincherle in ref. [3])

$$r_n = \frac{m_{n+1}}{m_n} = \frac{-b_{n+1}}{a_{n+1}-}\frac{b_{n+2}}{a_{n+2}-}\frac{b_{n+3}}{a_{n+3}-}\cdots \tag{12}$$

Any *minimal* solution is, up to a multiplication by a constant, *unique* [3]. This has two immediate consequences. First, $\phi_m(z)$ is, up to a multiplication constant, *unique*. Second, the ratio $r_0 = m_1/m_0$ of the first two terms of a given minimal solution is *unambiguously* fixed. Note that the ratio $r_0 = m_1/m_0$ involves m_0, although it takes into account the recurrence (2) only for $n \geq 1$ [3]. The remaining $n = 0$ part of the recurrence (2) imposes another condition [*cf.* eq. (10)] on the ratio $r_0 = m_1/m_0$. The condition (10) can be translated into the *boundary condition* on the logarithmic derivative of $\phi_m(z)$,

$$r_0 = \frac{\phi_m'(0)}{\phi_m(0)} = -a_0. \tag{13}$$

Obviously, one has $r_0 \neq -a_0$ in general. As a rule it is impossible to find a minimal solution satisfying an arbitrarily prescribed initial condition on the ratio m_1/m_0 of

its first two terms. (This point was not made clear by either Schweber [1] or later on by Braak [9].) If one defines the function $F = a_0 + r_0$, with r_0 given by the continued fraction in (12), the zeros of F would correspond to those points in a parameter space where the condition (13) is satisfied. Then the coefficients of $\phi_m(z) \in \mathcal{B}$ satisfy the recurrence (2) including $n = 0$, and $\phi_m(z)$ belongs to the spectrum. Analogous to the Schrödinger equation, the boundary condition (13) enforces quantization of energy levels.

The rest of the proof follows upon an application of the *Euler theorem*, which transforms the continued fraction defining r_0 in eq. (12) into an infinite series (*cf.* eqs. (4.4)-(4.5) on p. 43 of [3] forming the basis of the "*third*" method of Gautschi [3] at the special case of his $N = 0$ that is also used in our numerical implementation [11]). The latter leads directly to eqs. (8) and (9) for F [10].

Illustration of some basic properties. – In the case of a displaced harmonic oscillator, which is the special case of $\hat{H}_R$ in (7) for $\mu = 0$, the recurrence (2) becomes [*cf.* eq. (A.17) of ref. [1]]

$$c_{n+1} + \frac{n-x}{(n+1)\kappa}\, c_n + \frac{1}{n+1}\, c_{n-1} = 0, \tag{14}$$

where dimensionless parameter $\kappa = \lambda/\omega$ reflects the coupling strength and $x = \epsilon = E/\omega$ is a dimensionless energy parameter. Energy levels satisfy [1]

$$\epsilon_l = E_l/\omega = l - \kappa^2, \tag{15}$$

where $l \in \mathbb{N}$ is a *nonnegative* integer (including zero). As demonstrated in fig. 1, $F_{dho}(x)$ defined by eqs. (8) and (9) displays for $\omega = 1$ and $\kappa = 0.7$ a series of *discontinuous* branches extending monotonically between $-\infty$ and $+\infty$, which intersect the x-axis at $x_l = l - 0.49$, $l = 0, 1, \ldots$. The zeros correspond exactly to the position of the energy levels (15).

In the present example one can also explicitly illustrate that the difference equation (2) with the condition (10) imposed as an initial condition of the recurrence, and then solved upwardly, always possesses a solution. Obviously, such a solution would be generally a *dominant* solution. Consequently, the function $\phi(z)$ defined by power series expansion (3) with the expansion coefficients $\{d_j\}_{j=0}^{\infty}$ would exhibit (typically branch-cut [1]) *singularities* in the z-complex plane and would not belong to $\mathcal{B}$. Indeed, the recurrence (14) is solved for $n \geq 0$ with

$$c_n = \kappa^{\alpha-n} L_n^{(\alpha-n)}(\kappa^2), \tag{16}$$

where $\alpha = x + \kappa^2$ and L_n^{β} are associated Laguerre polynomials [19] (note different sign of κ compared to eq. (2.16) of Schweber [1]). The substitution (16) transforms (14) into a 3-point rule that is identically satisfied by the associated Laguerre polynomials [19]. The Rodrigues formula [19]

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} \frac{d^n}{dz^n} \left(e^{-z} z^{n+\alpha} \right) \tag{17}$$

implies $L_0^{(\alpha)}(z) = 1$, $L_1^{(\alpha)}(z) = -z + 1 + \alpha$, and in virtue of (16)

$$\begin{aligned} c_0 &= \kappa^{\alpha} L_0^{(\alpha)}(\kappa^2) = \kappa^{\alpha}, \\ c_1 &= \kappa^{\alpha-1} L_1^{(\alpha-1)}(\kappa^2) = \kappa^{\alpha-1} x. \end{aligned} \tag{18}$$

Thus $c_1/c_0 = x/\kappa$, which is exactly the $n = 0$ part of (14). In view of the asymptotic [19]

$$L_n^{(\alpha-n)}(z) \approx e^z \binom{\alpha}{n}, \qquad (n \to \infty) \tag{19}$$

where the (generalized) binomial coefficients

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}, \tag{20}$$

one finds from (16)

$$\frac{c_{n+1}}{c_n} \sim \frac{1}{\kappa} \frac{\alpha - n}{n+1} \to -\frac{1}{\kappa}, \qquad (n \to \infty) \tag{21}$$

whenever $\alpha \notin \mathbb{N}$. Therefore, unless α is a nonnegative integer, the solution of eq. (16) is the *dominant* solution, the power series solution (3) has only a *finite* radius of convergence and thus does not belong to $\mathcal{B}$. Only if the condition (15) is satisfied, $\alpha = x + \kappa^2$ is an *integer*, the *dominant* solution changes smoothly into the *minimal* solution of the recurrence (16), and the corresponding F_{dho} vanishes.

In the case of the Rabi model, the recurrence (2) becomes

$$c_{n+1} - \frac{f_n(x)}{(n+1)}\, c_n + \frac{1}{n+1}\, c_{n-1} = 0, \tag{22}$$

where $f_n(x)$ is given by

$$f_n(x) = 2\kappa + \frac{1}{2\kappa} \left(n - x - \frac{\Delta^2}{n-x} \right), \tag{23}$$

$\kappa = \lambda/\omega$, as in eq. (14), and dimensionless $\Delta = \mu/\omega$ (*cf.* eq. (A8) of Schweber [1], which has mistyped sign in front of his b_{n-1}, and eqs. (4) and (5) of [9]). At a first glance the recurrence (22) does not reduce to (14) for $\Delta = 0$ as one would expect from (7). The equivalence of eqs. (14) and (22) for $\Delta = 0$ is disguised by the fact that the recurrence (22) has been obtained after the unitary transformation of $\hat{H}_R$ in (7) induced by $D = \exp[\kappa(\hat{a}^\dagger - \hat{a})\sigma_1]$ [1]. Thereby the dimensionless energy parameter x in eqs. (22), (23) is $x = (E/\omega) + \kappa^2$.

One can verify that the entire function $\phi_m(z)$ generated by the *minimal* solution $\{m_j\}_{j=0}^{\infty}$ belongs to the Bargmann Hilbert space of analytical functions $\mathcal{B}$ for all model parameters, *i.e.* even if the quantization criterion $F = 0$ is not satisfied. Linearly independent solutions of recurrence (22) can be written in the limit $n \gg 1$ as

$$\nu_n = \begin{cases} 1/(n\kappa^n), & \text{dominant} \\ n^x \kappa^n / \Gamma(n+1), & \text{minimal.} \end{cases} \tag{24}$$

Let us consider the $z \to \infty$ asymptotic of

$$\chi(z) = \sum_{n=0}^{\infty} \nu_n(x) z^n \tag{25}$$

in the case of the minimal solution. Obviously, $\phi_m(z) \sim \chi(z)$ for $z \to \infty$. The asymptotic of $\chi(z)$ can be determined by the saddle point of the Euler-Maclaurin integral representation of $\chi(z)$ [21–23],

$$\chi(z) = \int_{\sigma}^{\infty} \nu(s,x) e^{s \ln z}\, ds + \mathcal{O}(|z|^{\sigma}) \qquad (z \to \infty), \tag{26}$$

σ being some constant, $-1 < \sigma < 0$ and

$$\nu(s,x) \approx \left(\frac{\kappa e}{s}\right)^s s^x = e^{s[\ln(\kappa e) - \ln s] + x \ln s} = e^{h(s)}. \tag{27}$$

For $z \to \infty$ and $|\arg z| \leq \pi/2$ one finds a unique saddle point in the cut complex plane that completely governs the asymptotic behaviour of $\chi(z)$ [21–23],

$$\chi(z) \sim \left(\frac{-2\pi}{h''(t_s)}\right)^{1/2} e^{\kappa z + x[1 + \ln(\kappa z)]} + \mathcal{O}(|z|^{\sigma}). \tag{28}$$

Here

$$h''(t_s) \approx -\frac{1}{\kappa z + x} - \frac{x}{(\kappa z + x)^2} \tag{29}$$

is the second derivative of $h(s)$ defined by eq. (27) at the saddle point $t_s \approx \kappa z + x$. Hence $\chi(z)$ behaves essentially as $e^{\kappa z}$ for $z \to \infty$. Such a behaviour of $\chi(z)$ was largely to be expected in view of that the asymptotic of the minimal solution (24) is only a slightly perturbed version of $1/\Gamma(n+1)$.

Discrete symmetries. – The Rabi Hamiltonian $\hat{H}_R$ [eq. (7)] is known to possess a discrete $\mathbb{Z}_2$-symmetry corresponding to the constant of motion, or parity, $\hat{P} = \exp(i\pi\hat{J})$ [4,9], where

$$\hat{J} = \hat{a}^\dagger \hat{a} + \frac{1}{2}(1 + \sigma_3) \tag{30}$$

is the familiar operator known to generate a continuous $U(1)$ symmetry of the JC model [8,9]. In contrast to ref. [9], our approach does not necessitate any active use of any underlying discrete symmetry. This has been demonstrated in a recent comment [10], where the regular spectrum of the Rabi model has been reproduced as zeros of a corresponding $F_{Rd}(x)$ based on the recurrence (22). Obviously it was not possible then to determine what is the parity of a state corresponding to a given zero of $F_{Rd}(x)$.

Nevertheless, not only any discrete symmetry can be easily incorporated in our approach, this can be accomplished more straightforwardly and more easy than in the approach of Braak [9]. Indeed, the Rabi Hamiltonian $\hat{H}_R$ is an example of a general Fulton and Gouterman Hamiltonian of a two-level system [20]

$$\hat{H}_{FG} = A\,\mathbf{1} + B\sigma_1 + C\sigma_3, \tag{31}$$

with

$$A = \omega \hat{a}^\dagger \hat{a}, \qquad B = \lambda(\hat{a}^\dagger + \hat{a}), \qquad C = \mu. \tag{32}$$

The Fulton and Gouterman symmetry operation $\hat{g}$ [20] is realized by *reflections*

$$\hat{a} \to -\hat{a}, \qquad \hat{a}^\dagger \to -\hat{a}^\dagger, \tag{33}$$

which leave the boson number operator $\hat{a}^\dagger\hat{a}$ invariant. Because $[\hat{g}, A] = [\hat{g}, C] = \{\hat{g}, B\} = 0$, $\hat{g}\sigma_3$ is the symmetry of $\hat{H}_R$ [20]. To any cyclic $\mathbb{Z}_2$ operator, such as $\hat{g}\sigma_3$, one can associate a pair of *projection operators*

$$P^{\pm} = \frac{1}{2}(1 \pm \hat{g}\sigma_3), \qquad \left(P^{\pm}\right)^2 = P^{\pm}. \tag{34}$$

However, because of $\hat{g}\sigma_3$, the projectors $P^{\pm}$ do not mix the upper and lower components of a wave function ϕ (conventional Pauli representation of σ_1 and σ_3 is assumed). In order to employ the Fulton and Gouterman reduction in the *positive* and *negative parity spaces*, wherein one component of ϕ is generated from the other by means of the operator $\hat{g}$, one is forced to work in a unitary equivalent *single-mode spin-boson picture*

$$\hat{H}_R = \omega \hat{a}^\dagger \hat{a} + \mu\sigma_1 + \lambda\sigma_3(\hat{a}^\dagger + \hat{a}). \tag{35}$$

The transformation is accomplished by means of the unitary operator $U = (\sigma_1 + \sigma_3)/\sqrt{2} = U^{-1}$. The transformation interchanges the expressions for B and C in (32), resulting in $\hat{g}\sigma_1$ becoming the symmetry of $\hat{H}_R$. Eqs. (4.12) and (4.13) of [20] then yield

$$[\omega \hat{a}^\dagger \hat{a} + \lambda(\hat{a}^\dagger + \hat{a}) \pm \mu\hat{g}]\phi^{\pm} = E^{\pm}\phi^{\pm}, \tag{36}$$

where the superscripts $\pm$ denote the positive and negative parity eigenstates of $P^{\pm}$ [with σ_3 being replaced by σ_1 in (34)]. Working in the Bargmann space,

$$[\omega z\partial_z + \lambda(z + \partial_z) \pm \mu\hat{g}]\phi^{\pm} = E^{\pm}\phi^{\pm}. \tag{37}$$

Eqs. (37) are equivalent to a coupled system of first-order eqs. (10) of the supplement to [9]. In contrast to [9], one does not need any ill motivated substitutions and vague arguments to arrive at the symmetry resolved spectrum of the Rabi model. Assuming power series expansions $\phi^{\pm}(z) = \sum_{n=0}^{\infty} c_n^{\pm} z^n$ for the positive and negative parity states, one arrives directly at the following three-term recurrence relations

$$c_{n+1}^{\pm} + \frac{1}{\kappa(n+1)}[n - x \pm (-1)^n\Delta]c_n^{\pm} + \frac{1}{n+1}\, c_{n-1}^{\pm} = 0, \tag{38}$$

where $x = E^{\pm}/\omega$ and $\kappa = \lambda/\omega$, as in eq. (14), and $\Delta = \mu/\omega$, as in eq. (22). In arriving at eqs. (38) we have merely used that

$$\hat{g}\phi^{\pm}(z) = \phi^{\pm}(-z) = \sum_{n=0}^{\infty} (-1)^n c_n^{\pm} z^n. \tag{39}$$

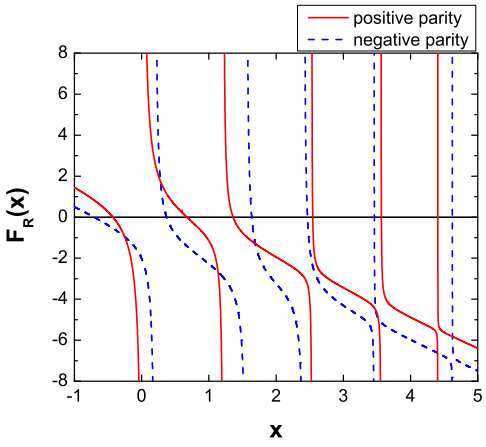

Fig. 2: $F_R(x)$ for $\kappa = 0.7$, $\Delta = 0.4$, and $\omega = 1$, *i.e.* the same parameters as in fig. 1 of ref. [9], shows zero at ≈ -0.707805 for a negative parity state and at ≈ -0.4270437 for a positive parity state. After addition of $\kappa^2 = 0.49$ they correspond to the zeros at -0.217805 and 0.0629563, respectively, in fig. 1 of refs. [9, 10].

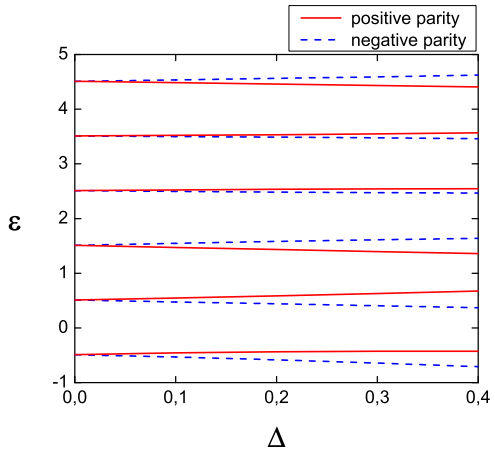

Fig. 3: Energy levels for $\kappa = 0.7$ and $\omega = 1$ as a function of Δ evolve from the doubly degenerate level corresponding to those of the displaced harmonic oscillator at $\epsilon_l = l - 0.49$ to the parity eigenstates of the Rabi model.

Upon defining corresponding $F_R^{\pm}(x)$ by means of eqs. (8) and (9), one can not only recover the regular spectrum of the Rabi, but also distinguish different parity eigenstates. This is demonstrated in fig. 2.

Fig. 3 demonstrates that on setting $\Delta = 0$ the parity eigenvalues become doubly-degenerate eigenvalues corresponding to those of a displaced harmonic oscillator. That was to be expected, because the three-term recurrence relations (38) reduce to the three-term recurrence (14) for $\Delta = 0$.

Discussion. – In his recent letter [9], Braak claimed to solve the Rabi model analytically (see also Viewpoint by Solano [24]). He suggested that a *regular* spectrum of the Rabi model was given by the zeros of transcendental functions $G_{\pm}(x)$ in the variable $x = (E/\omega) + \kappa^2$ [9],

$$G_{\pm}(x) = \sum_{n=0}^{\infty} K_n(x) \left[1 \mp \frac{\Delta}{x-n}\right] \kappa^n. \qquad (40)$$

Here the coefficients $K_n(x)$ were obtained recursively by solving the Poincaré difference equation (22) upwardly starting from the initial condition (10). Braak needed one page of arguments (between eq. (10) on p. 2 and the end of the first paragraph on p. 3 of the on-line supplement to [9]) to arrive at his $G_{\pm}(x)$. The arguments involved an ill motivated substitution and that a *sufficient* condition for the vanishing of an analytic function $G_+(x; z)$ (defined by eq. (16) of his supplement) for *all* $z \in \mathbb{C}$ is if it vanishes at a *single* point $z = 0$. The argument is essential to arrive at (40). However, as an example of any homogeneous polynomial shows, the argument is obviously invalid. Additionally, the coefficients K_n's in (40) at zeros of $G_{\pm}(x)$ have to be the minimal solutions. The difficulty of calculating minimal solutions can be illustrated by an attempt to calculate the Bessel functions of the first kind $J_n(x)$ for fixed $x = 1$ by an *upward* three-term recurrence from the initial values $J_0(x)$ and $J_1(x)$. It turns out that all digits calculated in single precision came out illusory already for $n \geq 7$ (*cf.* Table 1 of [3]).

We have advanced an alternative approach which does not require any explicit solution of any difference equation and which applies to an entire class of models. A symmetry is not required to characterize a quantum model in terms of a suitable $F(x)$. Contrary to ref. [9], $F(x)$ can be obtained directly from the recurrence coefficients characterizing an eigenvalue equation of a given model. In the case of the Rabi model, nothing but the elementary relation (39) has been employed in arriving from (37) to (38), and the corresponding $F_R^{\pm}(x)$ were unambiguously determined from (38) through eqs. (8) and (9). The resulting transcendental functions $F_R^{\pm}(x)$ are *different* from $G_{\pm}(x)$ considered by Braak [9]. With the exception of the displaced harmonic oscillator and the JC model, the zeros of $F(x)$ have to be still determined *numerically*. Therefore we disagree with Solano's view [24] that a mere characterization of a model by a function such as $G_{\pm}(x)$ of ref. [9], or our $F(x)$, is tantamount to solving the model analytically in a closed-form.

Our results rejuvenate the Schweber quantization criterion $r_0 + a_0 = 0$ known in the case when r_0 remained to be expressed in terms of continued fractions [*e.g.* eq. (12)] [1]. The Schweber criterion was deemed impractical and has not been employed to solve for the quantized energy levels of any quantum model [1, 4, 5, 9]. The criterion was either regarded to require a numerical diagonalization in a truncated Hilbert space (pp. 4-5 of the on-line Sup-

plement to ref. [9]), or simply refuted as being identically valid *irrespective* of the value of an energy parameter x (see for instance p. 4 of the on-line Supplement to ref. [9]). With the help of the Euler theorem, which transforms the continued fractions into an infinite series, our approach turned the Schweber quantization criterion into an efficient computational tool.

Our approach has been demonstrated on the examples of a displaced harmonic oscillator and the Rabi model. However, the outlined approach would work for any of the models of the class $\mathcal{R}$. For instance, the JC model and the single-mode spin-boson form of a generalized Rabi model introduced in [9],

$$H_{R\tau} = \omega\hat{a}^\dagger\hat{a} + \lambda\sigma_3(\hat{a}^\dagger + \hat{a}) + \tau\sigma_3 + \mu\sigma_1, \qquad (41)$$

where τ is a deformation parameter. Another example is a modified Rabi model with the interaction Hamiltonian $\hat{H}_{int} = i\hbar\lambda\sigma_1(\hat{a}^\dagger - \hat{a})$. The latter arises if the dipole interaction with a Fabry-Pérot cavity light mode is replaced by the interaction with a single-mode *plane-wave* field.

We have characterized our class $\mathcal{R}$ of models implicitly in that their eigenvalue equation can be reduced to a *difference equation* (2) with $\lim_{n\to\infty} a_n \neq 0$ and $\lim_{n\to\infty} b_n/a_n = 0$. An interesting problem is to provide an explicit characterization of the models.

Conclusions. – A general formalism has been developed which allows to determine the spectrum of an entire class of quantum models as zeros of a corresponding transcendental function $F(x)$. The function can be analytically determined as an infinite series defined solely in terms of recurrence coefficients. The class of quantum models comprises the displaced harmonic oscillator, the Jaynes-Cummings (JC) model, the Rabi model, and a generalized Rabi model. Applications of the Rabi model range from quantum optics and magnetic resonance to solid state and molecular physics. The model plays a prominent role in cavity QED and circuit QED, and can be experimentally realized in Cooper-pair boxes, flux q-bits, in Josephson junctions or using trapped ions, and is of importance for various approaches to quantum computing. Therefore, our results could have implications for further theoretical and experimental work that explores the interaction between light and matter, from weak to strong interactions. The ease in obtaining the spectrum is of importance regarding recent experimental advances in preparing (*ultra*)strongly interacting quantum systems, which can no longer be reliably described by the exactly solvable JC model. The relevant computer code has been made freely available online [11].

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